Angles in Elastic Two-Body Collisions

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Once upon a time in high school physics, I learned that the elastic collision between one object and a stationary object of the same mass results in both objects flying off at right angles to each other.

On a test soon after, there was a bonus question where we had to prove or disprove this outcome if the mass of the two objects were unequal.

Set-up

Variables

Let the first object be the moving one and the second object be the stationary one.

Without loss of generality, let m be the mass of the second object divided by the mass of the first object, i.e., $\frac{m_2}{m_1}$.

Let $\vec{c}$ be the initial velocity of the first object.
Let $\vec{d}$ be the initial velocity of the second object, which is zero by definition.
Let $\vec{a}$ be the final velocity of the first object.
Let $\vec{b}$ be the final velocity of the second object.

Note: For a vector $\vec{v}$, let v denote its magnitude (which would be explicitly written as $|\vec{v}|$).

Theorems

Conservation of momentum:

$$\vec{c} + m\vec{d} = \vec{a} + m\vec{b}$$

Conservation of energy, because this is an elastic collision:

$$c^2 + md^2 = a^2 + mb^2$$

Law of cosines:

$$z^2 = x^2 + y^2 - 2xy\cos Z$$

The proof

From the conservation of momentum, the vectors $\vec{c}$, $\vec{a}$, and $m\vec{b}$ form the sides of a triangle:

$$\vec{c} = \vec{a} + m\vec{b}$$

The conservation of energy and the law of cosines simultaneously hold:

$$c^2 = a^2 + mb^2$$

$$c^2 = a^2 + (mb)^2 - 2(a)(mb)\cos C$$

Subtract the previous equations and rearrange:

$$2(a)(mb)\cos C = (mb)^2 - mb^2$$

Use a little bit of magical girl power and get:

$$C = \cos^{-1}\frac{b(m-1)}{2a}$$

Now we note that $\cos^{-1}\frac{b(m-1)}{2a}$ is 90° iff $\frac{b(m-1)}{2a}$ is 0. This means one or more factors in the numerator is 0:

- If $\vec{b}$ is 0, then it implies that $\vec{c} = \vec{a}$ and no collision took place.
- If $m - 1$ is 0, then m is 1 and the two masses are equal.

Since it is not possible to make the argument of inverse cosine be 0 without breaking assumptions, the inverse cosine is not 90°.

Conclusion

Therefore, in a collision between one object and a stationary object of unequal mass, the two objects *never* leave at right angles to each other.

Other stuff

C is actually the angle between vectors $\vec{a}$ and $m\vec{b}$ when they are placed *head-to-tail*. The angle between these vectors when placed *tail-to-tail* is the supplementary angle, i.e., $180° - C$. But if C is $90°$, then $180° - C$ is $90°$ too.

The proof that a collision between an object and a stationary object of equal mass results in them leaving at right angles is left as an exercise for the reader.

The right angle property does not apply in the degenerate case where the first object comes to a complete stop and the second object picks up the full velocity.