The Poisson Distribution

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Description

In the Poisson distribution, events are independent and occur at a known average rate. Suppose we have an interval (of time, space, etc.) where the expected number of events is λ. Then the probability of exactly k events occuring in that interval is given by the following:

$$P(\lambda, k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Usage example

Problem

If 20 cars pass a checkpoint each minute (and the traffic follows a Poisson distribution), what is the probability of 2 cars passing the checkpoint in 1 second?

Solution

$k = 2$	The number of events we're looking for
$\lambda = 1/3$	Because $\frac{20 \text{ events}}{60 \text{ seconds}} = \frac{1/3 \text{ event}}{1 \text{ second}}$
$P(\lambda, k) = P\left(\frac{1}{3}, 2\right) = \frac{e^{-\frac{1}{3}} \cdot \frac{1}{3^2}}{2!} = \frac{e^{-\frac{1}{3}}}{18}$	Approximately 0.0398, or 4%

Derivation

Divide the interval in which events occur into n (discrete) slots. At each slot, the probability of an event occuring is λ/n.

By the binomial theorem, the probability of k events occuring in n trials where the success probability of each trial is λ/n is:

$$\binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$$

Because the Poisson distribution deals with events occuring on a continuous interval, let the number of slots be arbitrarily large. In other words, take the limit of that expression as n approaches infinity:

$$\begin{aligned}
P(\lambda, k) &= \lim_{n\to\infty}\binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \\
&= \lim_{n\to\infty}\binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-k} && \text{Expand} \\
&= \lim_{n\to\infty}\binom{n}{k}\left(\frac{\lambda}{n}\right)^k e^{-\lambda}\left(1-\frac{\lambda}{n}\right)^{-k} && \text{By the definition of } e \\
&= \lim_{n\to\infty}\binom{n}{k}\left(\frac{\lambda}{n}\right)^k e^{-\lambda} && \text{Because } 1-\frac{\lambda}{n} \text{ approaches } 1 \\
&= \lim_{n\to\infty}\frac{1}{n^k}\binom{n}{k}e^{-\lambda}\lambda^k && \text{Expand} \\
&= \lim_{n\to\infty}\frac{1}{n^k}\frac{1}{k!}\left(\prod_{m=n-k+1}^{n} m\right)e^{-\lambda}\lambda^k && \text{Express as a product} \\
&= \lim_{n\to\infty}\frac{1}{k!}\left(\prod_{m=n-k+1}^{n}\frac{m}{n}\right)e^{-\lambda}\lambda^k && \text{Put into product} \\
&= \lim_{n\to\infty}\frac{e^{-\lambda}\lambda^k}{k!} && \text{The product approaches } 1 \\
&= \frac{e^{-\lambda}\lambda^k}{k!} && \text{Limit of a constant}
\end{aligned}$$

Total probability

What is the total probability over all outcomes?

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} && \text{Sum over all outcomes} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} && \text{Extract constant out of summation} \\
&= e^{-\lambda} e^{\lambda} && \text{The sum is the power series for } e^{\lambda}\text{, valid for all } \lambda \in \mathbb{R} \\
&= e^0 && \text{By laws of exponents} \\
&= 1 && \text{By definition of exponential function}
\end{aligned}$$

As expected, the total probability is 1, regardless of the value of the parameter λ.